4. Computation Tree Logic (3)

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Outline

1. Reduced OBDD
2. Implementation of ROBDD-Based Algorithms
3. CTL*

1 Reduced OBDD

Definition 1 (Reduced OBDD)

Let $\mathfrak{B}$ be a $\wp$-OBDD. $\mathfrak{B}$ is called **reduced** if for every pair (v, w) of nodes in $\mathfrak{B}$:

$$v \neq w \implies f_v \neq f_w.$$

Let $\wp$-ROBDD denote a reduced $\wp$-OBDD.

- In reduced $\wp$-OBDDs any $\wp$-consistent cofactor is represented by exactly one node
- This is the key property to prove that reduced OBDDs yield a universal and canonical data structure for switching functions

1 Reduced OBDD

Theorem 2 (Universality and Canonicity of ROBDDs)

Let Var be a finite set of Boolean variables and $\wp$ a variable ordering for Var. Then:

1. *For each switching function f for Var there exists a $\wp$-ROBDD $\mathfrak{B}$ with $f_{\mathfrak{B}} = f$*
2. *Given two $\wp$-ROBDDs $\mathfrak{B}$ and $\mathfrak{C}$ with $f_{\mathfrak{B}} = f_{\mathfrak{C}}$, then $\mathfrak{B}$ and $\mathfrak{C}$ are isomorphic, i.e., agree up to renaming of the nodes*

Corollary 3 (Minimality of Reduced OBDDs)

Let $\mathfrak{B}$ be a $\wp$-OBDD for f. Then, $\mathfrak{B}$ is reduced if and only if $size(\mathfrak{B}) \leq size(\mathfrak{C})$ for each $\wp$-OBDD $\mathfrak{C}$ for f.

1.1 Reduction Rules

When the variable ordering $\wp$ for *Var* is fixed, then **reduced** $\wp$-OBDDs provide **unique** representations of switching functions for *Var*

1. **Elimination rule:** If v is an **inner node** of $\mathfrak{B}$ with

$$succ_0(v) = succ_1(v) = w,$$

   then remove v and redirect all incoming edges $u \rightsquigarrow v$ to w

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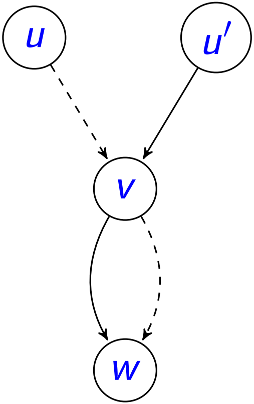
$$succ_0(v) = succ_1(v) = w,$$

   then remove v and redirect all incoming edges $u \rightsquigarrow v$ to w
2. **Isomorphism rule:** If v, w are nodes in $\mathfrak{B}$ with $v \neq w$ and
   - either v, w are drains (i.e., terminal nodes) with $val(v) = val(w)$ or
   - v, w are inner nodes with

$$\langle var(v), succ_1(v), succ_0(v) \rangle = \langle var(w), succ_1(w), succ_0(w) \rangle$$

   then remove node v and redirect all incoming edges $u \rightsquigarrow v$ to node w



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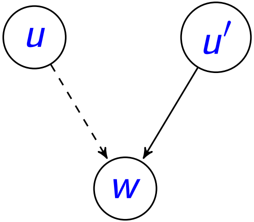

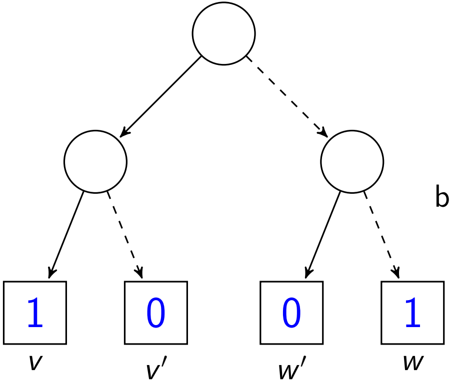

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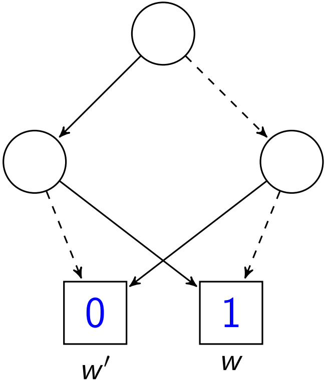
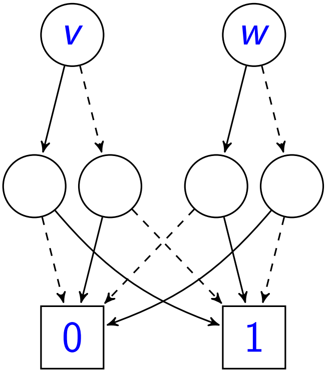


1.1 Reduction Rules



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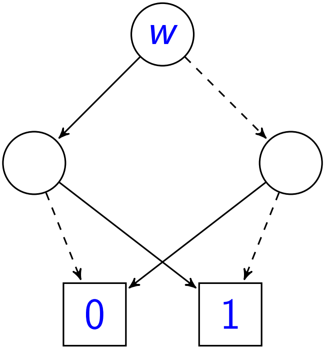


1.1 Reduction Rules

Both reduction rules are sound in the sense that they do not affect the semantics.

1. For the elimination rule applied to a z-node v with $w = succ_0(v) = succ_1(v)$, we have

$$f_v = (\neg z \wedge f_{succ_0(v)}) \wedge (z \wedge f_{succ_1(v)}) = (\neg z \wedge f_w) \wedge (z \wedge f_w) = f_w$$

2. If the isomorphism rule is applied to z-nodes v and w then

$$\begin{aligned} f_v &= (\neg z \wedge f_{succ_0(v)}) \wedge (z \wedge f_{succ_1(v)}) \\ &= (\neg z \wedge f_{succ_0(w)}) \wedge (z \wedge f_{succ_1(w)}) = f_w \end{aligned}$$

Theorem 4 (Completeness of Reduction Rules)

$\wp$-OBDD $\mathfrak{B}$ is reduced if and only if no reduction rule is applicable to $\mathfrak{B}$.

1 Reduced OBDD
- Reduction Rules
- Variable Ordering Problem

2 Implementation of ROBDD-Based Algorithms

3 CTL*

Let $m \geq 1$ and

$$f_m = (z_1 \wedge y_1) \vee (z_2 \wedge y_2) \vee \ldots \vee (z_m \wedge y_m).$$

1. For the variable ordering

$$\wp = (z_m, y_m, z_{m-1}, y_{m-1}, \ldots, z_1, y_1),$$

   the $\wp$-ROBDD for f_m has $2m + 2$ nodes,
2. while $\Omega(2^m)$ nodes are required for the ordering

$$\wp' = (z_1, z_2, \ldots, z_m, y_1, \ldots, y_m).$$

1.2 Variable Ordering Problem

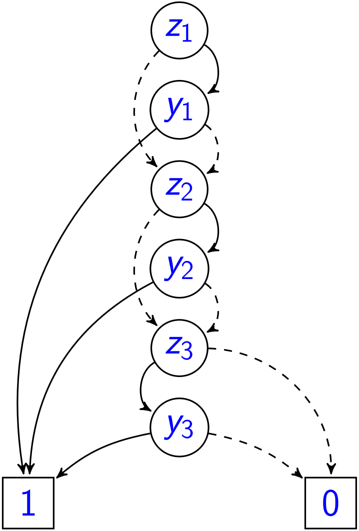


Figure: ROBDD for $f_3 = (z_1 \wedge y_1) \vee (z_2 \wedge y_2) \vee (z_3 \wedge y_3)$

1.2 Variable Ordering Problem

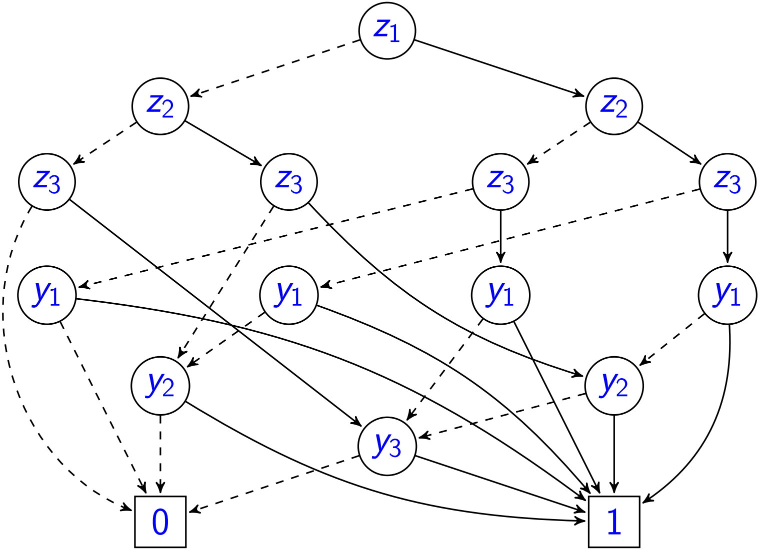


Figure: ROBDD for $f_3 = (z_1 \wedge y_1) \vee (z_2 \wedge y_2) \vee (z_3 \wedge y_3)$

1.2 Variable Ordering Problem

1. The size of ROBDDs is dependent on the variable ordering
2. To check whether a variable ordering is optimal is NP-hard
3. Many switching functions with large ROBDDs

Sifting algorithm

Dynamic variable ordering using variable swapping:

1. Select a variable x_i in OBDD
2. Successively swap x_i to determine $size(\mathfrak{B})$ at any position for x_i
3. Shift x_i to position for which $size(\mathfrak{B})$ is minimal
4. Go back to the first step until no improvement is made

1.2 Variable Ordering Problem

Symmetric Functions

$f \in Eval(z_1, ..., z_m 0)$ is **symmetric** if and only of

$$f([z_1 = b_1, ..., z_m = b_m]) = f([z_1 = b_{i_1}, ..., z_m = b_{i_m}])$$

for **each** permutation $(i_1, ..., i_m)$ of (1,...,m). Examples of symmetric functions:

1. $z_1 \vee z_2 \vee ... \vee z_m$
2. $z_1 \wedge z_2 \wedge ... \wedge z_m$
3. parity function, $z_1 \bigoplus z_2 \bigoplus ... \bigoplus z_m$, returns $\#(v \mapsto 1) = odd$
4. majority function, returns $\#(v \mapsto 1) > \#(v \mapsto 0)$

.

Lemma 5 (ROBDD-Size for Symmetric Functions)

If f is a symmetric function with m ***essential*** *variables, then for* ***each*** *variable ordering $\wp$ the $\wp$-ROBDD has size $O(m^2)$.*

1. Given a symmetric function f for m variables and $\wp = (z_1, \ldots, z_m)$,
2. then the $\wp$-consistent cofactors $f|z_1 = b_1, \ldots, z_i = b_i$ and $f|z_1 = c_1, \ldots, z_i = c_i$ agree for all bit tuples $(b_1, \ldots, b_i)$ and $(c_1, \ldots, c_i)$ that contain the same number of 1's.
3. Thus, there are at most $i + 1$ different $\wp$-consistent cofactors of f that arise by assigning values to the first i variables
4. Hence, the total number of $\wp$-consistent cofactors is bounded above by $\sum_{i=0}^{m}(i+1) = O(m^2)$ □

1.2 Variable Ordering Problem

ROBDDs vs. CNF/DNF

1. Given a CNF representation for f, the task to generate a CNF for $\neg f$ is expensive, CNF for $\neg f$ may have at least exponentially many clauses
2. However, negation is trivial as we may simply swap the values of the drains
3. For any variable ordering $\wp$, the $\wp$-ROBDDs for f and $\neg f$ have the same size
4. Regard the satisfiability problem, is NP-complete for CNF, but trivial for ROBDDs, since $f \neq 0$ if and only if the $\wp$-ROBDD for f does not contain a 0-drain
5. "linear time" for a ROBDD-based algorithm means linear in the size of the input ROBDDs

Outline

2 Implementation of ROBDD-Based Algorithms

The efficiency of ROBDD-based algorithms crucially relies on implementation techniques

1. Use a **single** reduced decision **graph** with **one global** variable ordering $\wp$ to represent several switching functions
2. All computations on these decision graphs are interleaved with the reduction rules to guarantee **redundance-freedom** at any time
3. The comparison of two represented **functions** simply requires checking equality of the **nodes** for them

1. Reduced OBDD

2. Implementation of ROBDD-Based Algorithms
   - Shared OBDD
   - Synthesizing shared ROBDDs

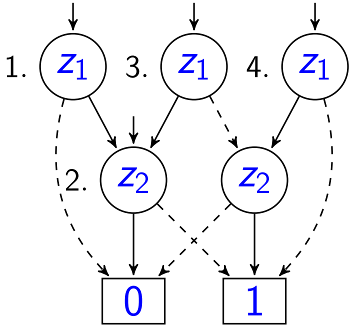
3. CTL*

2.1 Shared OBDD

Combine several OBDDs with same variable ordering such that common $\wp$-consistent co-factors are shared

Definition 6 (Shared OBDD)

- *Var*: a finite set of Boolean variables
- $\wp$ a variable ordering for Var
- **Shared** $\wp$-OBDD $\bar{\mathfrak{B}} = (V, V_I, V_T, succ_0, succ_1, var, val, \bar{v}_0)$
- $\bar{v}_0 = (v_0^1, \ldots, v_0^k)$ is a tuple of roots



$1.z_1 \wedge \neg z_2$, $2.\neg z_2$, $3.z_1 \oplus z_2$ and $4.\neg z_1 \vee z_2$

$\wp$-SOBDD is short for Shared $\wp$-OBDD

2.1 Shared OBDD

Symbolic CTL model-checking Φ:

1. the shared OBDD $\bar{\mathfrak{B}}$ with root nodes for Δ and
2. the f_a's, extended by new root nodes representing the characteristic functions of the satisfaction sets $Sat(\Psi)$, $\Psi \in Sub(\Phi)$
3. $\wp = (x_1, x'_1, ..., x_n, x'_n)$

2.2 Synthesizing shared ROBDDs

To support dynamic changes of the set of switching functions to be represented, two tables are required

1. **unique table:** contains the relevant information about the nodes and serves to keep the diagram reduced
2. **computed table:** for efficiency reasons

2.2 Synthesizing shared ROBDDs

The unique table

For each **inner node** v, the entries of the unique table are triples of the form

$$info(v) = \langle var(v), succ_1(v), succ_0(v) \rangle$$

1. contain the relevant information which is necessary for the applicability of the **isomorphism rule**
2. accessing via $op \triangleq find\_or\_add(z, v_1, v_0)$ with $v_1 \neq v_0$, and
3. return v **if there exists** a node $v = \langle z, v_1, v_0 \rangle$ in the ROBDD, or
4. create a **new** z-node v with $succ_0(v) = v_0$ and $succ_1(v) = v_1$
5. this op can be implemented using **hash functions**

ITE operator

ITE("if-then-else") operator for switching functions g, f_1, and f_2:

$$ITE(g, f_1, f_2) = (g \wedge f_1) \vee (\neg g \wedge f_2) \ .$$

1. ITE fits very well with the unique table by their info-triples

$$f_v = ITE(z, f_{succ_1(v)}, f_{succ_0 v}) \ .$$

2. $ITE(0, f_1, f_2) = f_2$, $ITE(1, f_1, f_2) = f_1$
3. $\neg f = ITE(f, 0, 1)$
4. $f_1 \vee f_2 = ITE(f_1, 1, f_2)$
5. $f_1 \wedge f_2 = ITE(f_1, f_2, 0)$
6. $f_1 \oplus f_2 = ITE(f_1, \neg f_2, f_2) = ITE(f_1, ITE(f_2, 0, 1), f_2)$
7. $f_1 \rightarrow f_2 = ITE(f_1, f_2, 1)$

Lemma 7 (Cofactors of ITE($\cdot$))

If g, f_1, f_2 are switching functions for Var, $z \in Var$ and $b \in \{0, 1\}$, then

$$ITE(g, f_1, f_2)|_{z=b} = ITE(g|_{z=b}, f_1|_{z=b}, f_2|_{z=b}) \ .$$

Proof: For any $(a, \bar{\mathfrak{c}}) = [z = a, \bar{y} = \bar{\mathfrak{c}}] \in Eval(Var)$

$$\begin{aligned}
& ITE(g, f_1, f_2)|_{z=b}(a, \bar{\mathfrak{c}}) \\
= \ & ITE(g, f_1, f_2)(b, \bar{\mathfrak{c}}) \\
= \ & (g(b, \bar{\mathfrak{c}}) \wedge f_1(b, \bar{\mathfrak{c}})) \vee (\neg g(b, \bar{\mathfrak{c}}) \wedge f_2(b, \bar{\mathfrak{c}})) \\
= \ & (g|_{z=b}(a, \bar{\mathfrak{c}}) \wedge f_1|_{z=b}(a, \bar{\mathfrak{c}})) \vee (\neg g|_{z=b}(a, \bar{\mathfrak{c}}) \wedge f_2|_{z=b}(a, \bar{\mathfrak{c}})) \\
= \ & ITE(g|_{z=b}, f_1|_{z=b}, f_2|_{z=b})(a, \bar{\mathfrak{c}}) \ .
\end{aligned}$$

$\square$

A node in a $\wp$-SOBDD for representing $ITE(g, f_1, f_2)$ is a node w such that $info(w) = \langle z, w_1, w_0 \rangle$ where

1. z is the minimal essential variable of $ITE(g, f_1, f_2)$ according to $<_\wp$
2. w_1, w_0 are SOBDD nodes with:

$$f_{w_1} = ITE(g|_{z=1}, f_1|_{z=1}, f_2|_{z=1})$$
$$f_{w_0} = ITE(g|_{z=0}, f_1|_{z=0}, f_2|_{z=0})$$

This suggests a recursive algorithm

1. determine $z = min\{var(u), var(v_1), var(v_2)\}$
2. recursively computes the nodes for ITE applied to the cofactors of $g = f_u, f_1 = f_{v_1}, f_2 = f_{v_2}$ for z

2.2 Synthesizing shared ROBDDs

Algorithm 1: $ITE(u, v_1, v_2)$(first version)

```
if u is terminal then
    if var(u) = 1 then
        w := v1;
    else
        w := v2;
    end
else
    z := min{var(u), var(v1), var(v2)};
    w1 := ITE(u|z=1, v1|z=1, v2|z=1);
    w0 := ITE(u|z=0, v1|z=0, v2|z=0);
    if w0 = w1 then
        w := w1;
    else
        w := find_or_add(z, w1, w0);
    end
end
return w;
```

Lemma 8 (ROBDD size of $ITE(g, f_1, f_2)$)

The size of the $\wp$-ROBDD for $ITE(g, f_1, f_2)$ is bounded by $N_g \cdot N_{f_1} \cdot N_{f_2}$ where N_f denotes the size of the $\wp$-ROBDD for f.

Proof: Let $\wp = (z_1, \ldots, z_m)$ and $\mathfrak{B}_f$ denote the $\wp$-ROBDD for f where the nodes are the $\wp$-consistent cofactors of f. The node set of $\mathfrak{B}_f$

$$V_f = \{f|_{z_1=b_1,\ldots,z_i=b_i} \mid 0 \leq i \leq m, b_1, \ldots, b_i \in \{0, 1\}\}$$

Note that several of the cofactors $f|z_1 = b_1, \ldots, z_i = b_i$ might agree.

Continue Proof:

- By Lemma 7, the node set $V_{ITE(g,f_1,f_2)}$ agrees with the set of switching functions

$$ITE(g|_{z_1=b_1,\ldots,z_i=b_i},\ f_1|_{z_1=b_1,\ldots,z_i=b_i},\ f_2|_{z_1=b_1,\ldots,z_i=b_i})$$

  where $0 \leq i \leq m$, $b_1, \ldots, b_i \in \{0, 1\}$.
- We can construct the surjective function

$$\iota : V_g \times V_{f_1} \times V_{f_2} \rightarrow V_{ITE(g,f_1,f_2)}.$$

  Hence,

$$\begin{aligned} N_{ITE(g,f_1,f_2)} &= |V_{ITE(g,f_1,f_2)}| \\ &\leq |V_g \times V_{f_1} \times V_{f_2}| = N_g \cdot N_{f_1} \cdot N_{f_2} \quad \square \end{aligned}$$

2.2 Synthesizing shared ROBDDs

Algorithm 2: $ITE(u, v_1, v_2)$(Use Computed Table)

if *there is an entry for* (u, v_1, v_2, w) *in the* ***computed table*** **then**
    return w;
else
    if u *is terminal* **then**
        if $var(u) = 1$ **then** $w := v_1$ **else** $w := v_2$;
    else
        $z := \min\{var(u), var(v_1), var(v_2)\}$;
        $w_1 := ITE(u|_{z=1}, v_1|_{z=1}, v_2|_{z=1})$;
        $w_0 := ITE(u|_{z=0}, v_1|_{z=0}, v_2|_{z=0})$;
        if $w_0 = w_1$ **then** $w := w_1$ **else** $w := find\_or\_add(z, w_1, w_0)$;
        insert (u, v_1, v_2, w) in the **computed table**;
    end
    return w;
end

Outline

1. **CTL*** is an extension of CTL as it allows path quantifiers $\exists$ and $\forall$ to be **arbitrarily nested with linear temporal operators** such as $\bigcirc$ and $\mathbf{U}$
   - $\forall\bigcirc\bigcirc a$ is a legal CTL* formula
   - but $\forall\bigcirc\bigcirc a$ is not in CTL
2. In contrast, in **CTL** each linear temporal operator must be **immediately preceded** by a path quantifier

CTL* **state formulae** Φ over AP, briefly called CTL* formulae,

$$\Phi ::= \mathit{true} \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \exists\varphi$$

where $a \in AP$ and φ is a **path formula**:

$$\varphi ::= \Phi \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \bigcirc\varphi \mid \varphi_1 \mathbf{U} \varphi_2$$

where Φ is a **state formula**, and φ, φ_1, and φ_2 are **path formulae**

$$\boxed{\Diamond\varphi = \mathit{true}\,\mathbf{U}\,\varphi, \quad \Box\varphi = \neg\Diamond\neg\varphi, \quad \forall\varphi = \neg\exists\neg\varphi}$$

The satisfaction relation $\models$ is defined for state formulae by

$$
\begin{array}{lll}
s \models a & \text{iff} & a \in L(s), \\
s \models \neg\Phi & \text{iff} & \text{not } s \models \Phi, \\
s \models \Phi \wedge \Psi & \text{iff} & (s \models \Phi) \text{ and } (s \models \Psi) \\
s \models \exists\varphi & \text{iff} & \pi \models \varphi \text{ for some } \pi \in Paths(s).
\end{array}
$$

For path $\pi = s_0 s_1 s_2 \ldots$, the satisfaction relation $\models$ for path formulae is defined by:

$$
\begin{array}{lll}
\pi \models \Phi & \text{iff} & s_0 \models \Phi, \\
\pi \models \varphi_1 \wedge \varphi_2 & \text{iff} & (\pi \models \varphi_1) \text{ and } (\pi \models \varphi_2), \\
\pi \models \neg\varphi & \text{iff} & \pi \not\models \varphi, \\
\pi \models \bigcirc\varphi & \text{iff} & \pi[1..] \models \varphi, \\
\pi \models \varphi_1 \mathbf{U} \varphi_2 & \text{iff} & \exists j \geq 0.\ (\pi[j..] \models \varphi_2 \wedge \\
 & & \quad (\forall 0 \leq k < j.\ \pi[k..] \models \varphi_1))
\end{array}
$$

- $\pi[i..]$ denotes the suffix of π from index $i \geq 0$ on

For CTL*-state formula Φ, the satisfaction set $Sat(\Phi)$ is defined by

$$Sat(\Phi) = \{s \in S \mid s \models \Phi\} .$$

The transition system TS satisfies CTL* formula Φ if and only if Φ holds in **all initial states** of TS:

$$TS \models \Phi \quad \text{if and only if} \quad \forall s_0 \in I.\ s_0 \models \Phi .$$

3.4 Embedding of LTL in CTL*

For each LTL formula φ over AP and for each $s \in S$:

$$\underbrace{s \models \varphi}_{\text{LTL semantics}} \quad \text{iff} \quad \underbrace{s \models \forall\varphi}_{\text{CTL* semantics}}$$

$$\underbrace{TS \models \varphi}_{\text{LTL semantics}} \quad \text{iff} \quad \underbrace{TS \models \forall\varphi}_{\text{CTL* semantics}}$$

It is justified to understand LTL as a sublogic of CTL*

Theorem 9 (CTL* is More Expressive Than LTL and CTL)

For the CTL formula over* $AP = \{a, b\}$,

$$\Phi = (\forall\Diamond\Box a) \vee (\forall\Box\exists\Diamond b) \ ,$$

there does not exist any equivalent LTL or CTL formula

Proof:

1. $\forall\Box\exists\Diamond b$ is a CTL formula but not in LTL
2. $\Diamond\Box a$ is an LTL formula but not in CTL

$\Box$

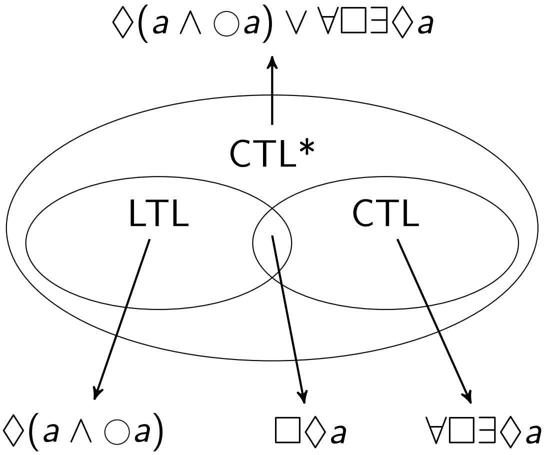


Figure: Relationship between LTL, CTL and CTL* duality laws for path quantifiers

$$\neg\forall\varphi \equiv \exists\neg\varphi$$
$$\neg\exists\varphi \equiv \forall\neg\varphi$$

distributive laws

$$\forall(\varphi_1 \wedge \varphi_2) \equiv \forall\varphi_1 \wedge \forall\varphi_2$$
$$\exists(\varphi_1 \vee \varphi_2) \equiv \exists\varphi_1 \vee \exists\varphi_2$$

quantifier absorption laws

$$\forall\Box\Diamond\varphi \equiv \forall\Box\forall\Diamond\varphi$$
$$\exists\Diamond\Box\varphi \equiv \exists\Diamond\exists\Box\varphi$$

CTL$^+$ extends CTL by allowing Boolean operators in path formulae:

1. CTL$^+$ state formulae Φ

$$\Phi ::= \mathit{true} \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \exists\varphi \mid \forall\varphi$$

2. CTL$^+$ path formulae φ

$$\varphi ::= \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \bigcirc\Phi \mid \Phi_1 \mathbf{U} \Phi_2$$

3. CTL$^+$ is as expressive as CTL

For any CTL$^+$ state formula Φ^+ there exists an equivalent CTL formula Φ. For example:

$$\underbrace{\exists(a\,\mathbf{W}\,b)}_{\text{CTL formula}} \equiv \underbrace{\exists((a\,\mathbf{U}\,b) \vee \Box a)}_{\text{CTL}^+\text{ formula}}$$

$$\underbrace{\exists(\Diamond a \wedge \Diamond b)}_{\text{CTL}^+\text{ formula}} \equiv \underbrace{\exists\Diamond(a \wedge \exists\Diamond b) \wedge \exists\Diamond(b \wedge \exists a)}_{\text{CTL formula}}$$

The transformation relies on equivalence laws such as

$$\begin{aligned}
\exists(\neg\bigcirc\Phi) &\equiv \exists\bigcirc\neg\Phi \\
\exists(\neg(\Phi_1\,\mathbf{U}\,\Phi_2)) &\equiv \exists((\Phi_1 \wedge \neg\Phi_2)\,\mathbf{U}\,(\neg\Phi_1 \wedge \neg\Phi_2)) \vee \exists\Box\neg\Phi_2 \\
\exists(\bigcirc\Phi_1 \wedge \bigcirc\Phi_2) &\equiv \exists\bigcirc(\Phi_1 \wedge \Phi_2) \\
&\vdots
\end{aligned}$$

1. CTL can be expanded by means of a Boolean operator for path formulae without changing the expressiveness
2. CTL$^+$ formulae can be much shorter than the shortest equivalent CTL formulae

The model-checking algorithm for CTL* is based on a bottom-up traversal of the syntax tree of Φ

Definition 10 (Maximal Proper State Subformula)

State formula Ψ is a maximal proper state subformula of Φ whenever

1. Ψ is a subformula of that differs from Φ
2. and Ψ is not contained in any other proper state subformula of Φ

Recursive computation of satisfaction sets:

$$
\begin{aligned}
Sat(true) &= S \\
Sat(a) &= \{s \in S \mid a \in L(s)\} \\
Sat(\Phi_1 \wedge \Phi_2) &= Sat(\Phi_1) \cap Sat(\Phi_2) \\
Sat(\neg\Phi) &= S \setminus Sat(\Phi) \\
Sat(\forall\varphi) &= Sat_{LTL}(\varphi) \\
Sat(\exists\varphi) &= S \setminus Sat_{LTL}(\neg\varphi)
\end{aligned}
$$

Φ_1 and Φ_2 are maximal proper state subformulae of Φ:

$$\Phi = \underbrace{\exists\Diamond\Box a}_{\Phi_1} \wedge \exists\Box(\bigcirc b \wedge \Diamond \underbrace{\neg\exists(a\,\mathbf{U}\,b)}_{\Phi_2})$$

1. calculate recursively the satisfaction sets $Sat(\Phi_i)$
2. replace Φ_i with the atomic proposition a_i, $i = 1, 2$

$$\Phi \rightsquigarrow a_1 \wedge \exists \underbrace{\Box(\bigcirc b \wedge \Diamond a_2)}_{\text{LTL formula } \varphi} = a_1 \wedge \exists\varphi$$

3. compute $Sat(\exists\varphi)$ via NBA $\mathcal{A}$ for φ and nested DFS in $TS \oplus \mathcal{A}$
4. return $Sat(\Phi) = Sat(a_1 \wedge \exists\varphi) = Sat(a_1) \cap Sat(\exists\varphi)$

3.7 CTL* Model Checking

Algorithm 3: CTL* model checking algorithm (basic idea)

Input: TS with initial states I, and CTL* formula Φ
Output: $I \subseteq Sat(\Phi)$

foreach $i \leq |\Phi|$ **do**
  foreach $\Psi \in Sub(\Phi)$ *with* $|\Psi| = i$ **do**
    switch Ψ **do**
      case *true* **do** $Sat(\Psi) := S$;
      case a **do** $Sat(\Psi) := \{s \in S \mid a \in L(s)\}$;
      case $a_1 \wedge a_2$ **do** $Sat(\Psi) := Sat(a_1) \cap Sat(a_2)$;
      case $\neg a$ **do** $Sat(\Psi) := S \setminus Sat(a)$;
      case $\exists\varphi$ **do** $Sat(\Psi) := S \setminus Sat_{LTL}(\neg\varphi)$;
    end
    $AP := AP \cup \{a_\Psi\}$;
    replace Ψ with a_Ψ;
    foreach $s \in Sat(\Psi)$ **do** $L(s) := L(s) \cup \{a_\Psi\}$;
  end
end
return $I \subseteq Sat(\Phi)$;

3.7 CTL* Model Checking

Evidently, the time complexity of the CTL* model-checking algorithm is dominated by the LTL model-checking phases

Theorem 11 (Time Complexity of CTL* Model Checking)

For transition system TS with N states and K transitions, and CTL formula Φ, the CTL* model-checking problem $TS \models \Phi$ can be determined in time $O((N + K) \cdot 2^{|\Phi|})$.*

- CTL* model checking can be solved by any LTL model-checking algorithm
- The CTL* model-checking problem is PSPACE-complete

Summary

1. CTL is a logic for formalizing properties over computation trees, i.e., the branching structure of the states
2. The expressivenesses of LTL and CTL are incomparable
3. The CTL model-checking problem can be solved by a recursive descent procedure over the parse tree of the state formula to be checked
4. The time complexity of the CTL model-checking algorithm is linear in the size of the transition system and the length of the formula
5. The CTL model-checking procedure can be realized symbolically by means of ordered binary decision diagrams
6. CTL* is more expressive than either CTL and LTL
7. The CTL* model-checking problem can be solved by an appropriate combination of the recursive descent procedure (as for CTL) and the LTL model-checking algorithm, PSPACE-complete